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Rigid Motions in the Presence of a Magnetic Field

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It is shown that, in the standard framework of non-relativistic quantum mechanics, the presence of a magnetic field implies that there are no operators representing those translations or rotations that do not leave invariant the magnetic field, and the corresponding components of the linear or angular momentum are undefined.

KEY WORDS: quantum mechanics; magnetic field; groups of motions.

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1. INTRODUCTION

The usual Hamiltonian for a charged particle in an external magnetic field involves, not the magnetic field itself, but the vector potential, which is gauge dependent and may not possess the symmetries of the magnetic field. For instance, a uniform magnetic field is invariant under all translations but a vector potential invariant under all translations can only yield a vanishing magnetic field. This fact might seem strange, taking into account the relationship between symmetries and conserved quantities and that the equations of motion do not contain the vector potential. The aim of this paper is to show that, for a charged particle in a magnetic field, an operator representing a rigid translation or rotation exists if and only if the magnetic field is invariant under that transformation and that one has to distinguish between the kinematical momentum, the canonical momentum, and the infinitesimal generator of translations, which exists if and only if the magnetic field is invariant under continuous translations. A similar result holds in classical mechanics where a rigid motion is a canonical transformation if and only if the magnetic field is invariant under that transformation (Torres del Castillo, 2005). We also show that, in the presence of a magnetic field, the wave function cannot be an ordinary function but a section of a line bundle. Some of the results presented

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here have been previously obtained, especially in connection with electrons in a periodic potential, without identifying their origin in the very definition of the operators that represent rigid motions (see, e.g., Brown, 1964; Dereli and Verçin, 1993; Zack, 1964).

2. MAGNETIC FIELDS AND SYMMETRIES

The usual Hamiltonian for a charged particle of mass m and charge q in an external magnetic field is given by

$$H = \frac{\mathbf{p}^2}{2m} = \frac{1}{2m}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right)^2, \tag{1}$$

where $\mathbf{A}$ is a vector potential for the magnetic field, $\mathbf{p} = m\mathbf{v}$ is the *kinematical* momentum and

$$\mathbf{P} = \mathbf{p} + \frac{q}{c}\mathbf{A} \tag{2}$$

is the *canonical* momentum, which, together with the position operator $\mathbf{x}$, satisfies the commutation relations

$$[x_i, x_j] = 0, \quad [x_i, P_j] = i\hbar\delta_{ij}, \quad [P_i, P_j] = 0, \tag{3}$$

where the x_i and P_i are the Cartesian components of $\mathbf{x}$ and $\mathbf{P}$, respectively. These relations imply that the canonical momentum operators can be represented as

$$P_k = \frac{\hbar}{i}\frac{\partial}{\partial x_k} \tag{4}$$

and that

$$[p_i, p_j] = i\hbar\frac{q}{c}\varepsilon_{ijk}B_k \tag{5}$$

(see, e.g., Sakurai, 1994). If $T(\mathbf{a})$ is the operator representing a rigid translation by a constant vector $\mathbf{a}$ then we should have

$$T(\mathbf{a})^{-1}x_i T(\mathbf{a}) = x_i + a_i, \quad T(\mathbf{a})^{-1}p_i T(\mathbf{a}) = p_i. \tag{6}$$

The second of these relations, which is crucial in what follows, is the right condition to impose since the Cartesian components of the velocities should not be affected by a rigid translation. Hence, from Equations (5) and (6) we obtain

$$\begin{aligned} i\hbar\frac{q}{c}\varepsilon_{ijk}T(\mathbf{a})^{-1}B_k T(\mathbf{a}) &= T(\mathbf{a})^{-1}[p_i, p_j]T(\mathbf{a}) \\ &= [p_i, p_j] \\ &= i\hbar\frac{q}{c}\varepsilon_{ijk}B_k, \end{aligned}$$

which means that $\mathbf{B}$ must be invariant under the translation represented by $T(\mathbf{a})$. Thus, the existence of the operator $T(\mathbf{a})$, representing a translation by $\mathbf{a}$, requires the invariance of the magnetic field under that translation. It should be remarked that this result does not depend on which is the complete expression of the Hamiltonian and it applies for discrete or continuous translations.

Denoting by $\mathcal{P}_k$ the infinitesimal generator of translations along the x_k-axis, that is, to first order in $\Delta\mathbf{x}$

$$T(\Delta\mathbf{x}) \simeq 1 - \frac{i}{\hbar}\Delta\mathbf{x}\cdot\boldsymbol{\mathcal{P}}, \tag{7}$$

from Equations (6) we obtain the commutation relations

$$[x_i, \mathcal{P}_j] = i\hbar\delta_{ij}, \quad [p_i, \mathcal{P}_j] = 0. \tag{8}$$

From the discussion above, it follows that $\mathcal{P}_k$ exists only if the magnetic field is invariant under continuous translations along the x_k-axis; in fact, making use of the Jacobi identity and Equations (5) and (8) we find that

$$\begin{aligned} 0 &= [[p_i, p_j], \mathcal{P}_k] + [[\mathcal{P}_k, p_i], p_j] + [[p_j, \mathcal{P}_k], p_i] \\ &= i\hbar\frac{q}{c}\varepsilon_{ijs}[B_s, \mathcal{P}_k] \\ &= (i\hbar)^2\frac{q}{c}\varepsilon_{ijs}\frac{\partial B_s}{\partial x_k}, \end{aligned}$$

which is equivalent to $\partial B_s/\partial x_k = 0$ $(s = 1, 2, 3)$. (Cf., however, Jackiw, 2004 where the kinematical momentum is taken as the infinitesimal generator of translations.) Furthermore, from the first equality in Equation (1) it follows that if there are no other interactions, whenever exists, $\mathcal{P}_k$ *commutes* with the Hamiltonian (see the example below).

In a similar manner one can show that an operator representing a rigid rotation exists if and only if the magnetic field is invariant under that rotation and that, if the operator $\mathcal{L}_i$ is the infinitesimal generator of rotations about the x_i-axis, it must satisfy the commutation relations

$$[x_i, \mathcal{L}_j] = i\hbar\varepsilon_{ijk}x_k, \quad [p_i, \mathcal{L}_j] = i\hbar\varepsilon_{ijk}p_k, \tag{9}$$

hence, by virtue of the Jacobi identity

$$\begin{aligned} 0 &= [[p_i, p_j], \mathcal{L}_k] + [[\mathcal{L}_k, p_i], p_j] + [[p_j, \mathcal{L}_k], p_i] \\ &= i\hbar\left(\frac{q}{c}\varepsilon_{ijl}[B_l, \mathcal{L}_k] - \varepsilon_{ikl}[p_l, p_j] + \varepsilon_{jkl}[p_l, p_i]\right) \\ &= (i\hbar)^2\frac{q}{c}\varepsilon_{ijl}\left(\varepsilon_{krm}x_r\frac{\partial B_l}{\partial x_m} - \varepsilon_{kml}B_m\right), \end{aligned}$$

which means that the magnetic field must be invariant under the rotations about the x_k-axis in order for $\mathcal{L}_k$ to exist. ($\varepsilon_{krm}x_r\partial B_l/\partial x_m - \varepsilon_{kml}B_m$ are the Cartesian components of the Lie derivative of $\mathbf{B}$ with respect to $\varepsilon_{krm}x_r\partial/\partial x_m$.) The explicit expressions for $\mathcal{P}_k$ and $\mathcal{L}_k$ depend on that of $\mathbf{B}$.

2.1. Uniform Magnetic Field

As a first example we shall consider the case of a static uniform magnetic field $\mathbf{B} = (0, 0, B)$, which is invariant under all rigid translations and the rotations about the x_3-axis. Making use of Equations (2), (3), and (8) one finds that

$$\mathcal{P}_1 = p_1 - \frac{qB}{c}x_2, \quad \mathcal{P}_2 = p_2 + \frac{qB}{c}x_1, \quad \mathcal{P}_3 = p_3, \tag{10}$$

are infinitesimal generators of translations along the Cartesian axes (without having to choose a gauge), i.e.,

$$\mathcal{P}_i = p_i - \frac{q}{c}\varepsilon_{ijk}x_j B_k.$$

It may be noticed that $\mathcal{P}_1$ does not commute with $\mathcal{P}_2$; in fact, $[\mathcal{P}_1, \mathcal{P}_2] = -i\hbar qB/c$. Thus, in spite of being the infinitesimal generators of translations, $\mathcal{P}_1$ and $\mathcal{P}_2$ do not commute with each other, but their commutator is a constant. This fact does not lead to inconsistencies with the assumed behavior of the observables under translations, since Equations (6) are invariant under the multiplication of $T(\mathbf{a})$ by any complex factor of unit modulus (see below).

Among other things, this means that $\mathcal{P}_1$ and $\mathcal{P}_2$ *cannot* be represented by the differential operators $-i\hbar\partial/\partial x_1$ and $-i\hbar\partial/\partial x_2$, respectively, and also leads to the conclusion that the wave function ψ cannot be an ordinary complex-valued function, since for an ordinary function Taylor's formula

$$\psi(\mathbf{x} - \mathbf{a}) \simeq \psi(\mathbf{x}) - \frac{i\mathbf{a}}{\hbar}\frac{\hbar}{i} \cdot \nabla\psi(\mathbf{x}),$$

implies that the infinitesimal generator of translations is $-i\hbar\nabla$. Instead, ψ is the *local* expression of a *section* of a line bundle. (Roughly speaking, ψ is the only component of a vector field whose values belong to a complex one-dimensional (internal) vector space. For a precise definition see, e.g., Ward and Wells, 1990.)

This conclusion is not new, and is related to the fact that the electromagnetic interaction can be regarded as a gauge theory of the Yang–Mills type. The point to be stressed here is that the results above show that it is necessary to treat the wave function as a section of a line bundle.

One can verify directly that the three momenta given by Equations (10) are indeed conserved if H is given by Equation (1); the momentum $(qB/c)(-x_2, x_1, 0)$, added to the kinematical momentum of the charged particle, is the momentum of the combined electromagnetic field, which cannot be computed using the standard expression $\int \mathbf{E} \times \mathbf{B}\, dv/(4\pi c)$ (see, e.g., Jackson, 1975) that is not applicable in this case since the field does not tend to zero at infinity (cf. also Jáuregui and Hacyan, 2005).

Noting that, for a pair of constant vectors, **a**, **b**,

$$[\mathbf{a}\cdot\mathcal{P}, \mathbf{b}\cdot\mathcal{P}] = -i\hbar\frac{q}{c}\mathbf{a}\times\mathbf{b}\cdot\mathbf{B}$$

one finds that

$$\exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\mathcal{P}\right)\exp\left(-\frac{i}{\hbar}\mathbf{b}\cdot\mathcal{P}\right) = \exp\left(\frac{iq}{2\hbar c}\mathbf{a}\times\mathbf{b}\cdot\mathbf{B}\right)\exp\left[-\frac{i}{\hbar}(\mathbf{a}+\mathbf{b})\cdot\mathcal{P}\right]$$

and therefore

$$\begin{aligned}&\exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\mathcal{P}\right)\exp\left(-\frac{i}{\hbar}\mathbf{b}\cdot\mathcal{P}\right)\\ &\quad= \exp\left(\frac{iq}{\hbar c}\mathbf{a}\times\mathbf{b}\cdot\mathbf{B}\right)\exp\left(-\frac{i}{\hbar}\mathbf{b}\cdot\mathcal{P}\right)\exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\mathcal{P}\right)\end{aligned}$$

which explicitly shows that, as far as the operators **x** and **p** is concerned, $\exp(-i\mathbf{a}\cdot\mathcal{P}/\hbar)$ corresponds to a translation along **a**; however, the wave function acquires a phase factor when the particle is translated along a closed rectangle (see also Brown, 1964).

According to Equations (2) and (4)

$$p_i = -i\hbar\frac{\partial}{\partial x_i} - \frac{q}{c}A_i, \tag{11}$$

therefore, if we choose

$$\mathbf{A} = B(-x_2, 0, 0), \tag{12}$$

from Equations (10) we obtain in this case

$$\begin{aligned}\mathcal{P}_1 &= -i\hbar\frac{\partial}{\partial x_1},\\ \mathcal{P}_2 &= -i\hbar\frac{\partial}{\partial x_2} + \frac{qB}{c}x_1,\\ \mathcal{P}_3 &= -i\hbar\frac{\partial}{\partial x_3}.\end{aligned} \tag{13}$$

Similarly, from Equations (9) one finds that an infinitesimal generator of rotations about the x_3-axis is

$$\mathcal{L}_3 = x_1p_2 - x_2p_1 + \frac{qB}{2c}\left(x_1^2 + x_2^2\right). \tag{14}$$

The term $(qB/2c)(x_1^2 + x_2^2)$ corresponds to the x_3-component of the angular momentum of the electromagnetic field. Assuming that the vector potential is given by Equation (12), the expression (14) reduces to

$$\mathcal{L}_3 = -i\hbar \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) + \frac{qB}{2c} \left(x_1^2 - x_2^2 \right).$$

By combining Equations (1) and (10) one can write the Hamiltonian (1) in terms of the $\mathcal{P}_i$ as

$$H = \frac{1}{2m} \left[\left(\mathcal{P}_1 + \frac{qB}{c} x_2 \right)^2 + \left(\mathcal{P}_2 - \frac{qB}{c} x_1 \right)^2 + \mathcal{P}_3^2 \right].$$

Since each $\mathcal{P}_i$ commutes with H, and $\mathcal{P}_1$ commutes with $\mathcal{P}_3$ [see Equation (13)], there exist common eigenfunctions of H, $\mathcal{P}_1$, and $\mathcal{P}_3$. Making use of the explicit expressions (13) we see that an eigenfunction of H, $\mathcal{P}_1$, and $\mathcal{P}_3$ with eigenvalues E, π_1, and π_3, respectively, is of the form

$$\psi = f(x_2) \exp \left[\frac{i}{\hbar} (\pi_1 x_1 + \pi_3 x_3) \right], \tag{15}$$

where f obeys the equation

$$-\frac{\hbar^2}{2m} \frac{d^2 f}{dx_2^2} + \frac{q^2 B^2}{2mc^2} \left(x_2 + \frac{\pi_1 c}{qB} \right)^2 f = \left(E - \frac{\pi_3^2}{2m} \right) f, \tag{16}$$

which has the form of the Schrödinger equation for a one-dimensional harmonic oscillator (cf. Brown, 1964; Landau and Lifshitz, 1999).

Following the standard approach, one would write the Hamiltonian (1) in the form

$$H = -\frac{\hbar^2}{2m} \left[\left(\frac{\partial}{\partial x_1} + \frac{i}{\hbar} \frac{qB}{c} x_2 \right)^2 + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right]$$

[see Equation (12)] and, since x_1 and x_3 are ignorable coordinates, one can look for separable eigenfunctions of H of the form (15), with the function f determined by Equation (16). Thus, the eigenfunctions and eigenvalues of H are the same, regardless of the interpretation of the operators.

2.2. Piecewise Uniform Magnetic Field

The vector potential

$$\mathbf{A} = \begin{cases} \frac{1}{2} B(-x_2, x_1, 0) & \text{for } x_1^2 + x_2^2 \leq a^2, \\ \frac{1}{2} B a^2 \dfrac{(-x_2, x_1, 0)}{x_1^2 + x_2^2} & \text{for } x_1^2 + x_2^2 > a^2, \end{cases} \tag{17}$$

where a is some positive constant with dimension of length, is continuous and yields a uniform magnetic field $\mathbf{B} = (0, 0, B)$ inside the circular cylinder

$x_1^2 + x_2^2 = a^2$ and a magnetic field equal to zero outside the cylinder. Now the magnetic field is invariant under translations along the x_3-axis and rotations about this axis and therefore only $\mathcal{P}_3$ and $\mathcal{L}_3$ are defined. According to our previous results we have [see Equations (2), (10), and (14)] $\mathcal{P}_3 = -i\hbar\partial/\partial x_3$ and

$$\mathcal{L}_3 = \begin{cases} -i\hbar\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) & \text{for } x_1^2 + x_2^2 \leq a^2, \\ -i\hbar\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) - \frac{qBa^2}{2c} & \text{for } x_1^2 + x_2^2 > a^2. \end{cases} \tag{18}$$

The constant term $-qBa^2/(2c)$ in the last expression can be eliminated on the basis that Equations (9) define $\mathcal{L}_3$ up to a constant multiple of the identity operator and that in this way we obtain a continuous expression for $\mathcal{L}_3$. (Note that a similar procedure is not necessary or useful in the case considered in the foregoing subsection.)

2.3. Magnetic Monopole

Another illustrative example is given by the field of a magnetic monopole, even though, as is well-known, it raises some problems since

$$\begin{aligned}[p_1, [p_2, p_3]] + [p_2, [p_3, p_1]] + [p_3, [p_1, p_2]] &= i\hbar\frac{q}{c}[p_j, B_j] \\ &= \frac{\hbar^2 q}{c}\nabla\cdot\mathbf{B}\end{aligned}$$

does not vanish at the monopole's position, by contrast with what is required by the Jacobi identity (see also Jackiw, 1980, 2004).

The magnetic field of a monopole of magnetic charge g placed at the origin would be

$$\mathbf{B} = \frac{g\mathbf{x}}{|\mathbf{x}|^3}, \tag{19}$$

which is invariant under rotations about any axis passing through the origin and therefore the three generators of rotations about the Cartesian axes are well defined [and commute with the Hamiltonian (1)].

Making use of Equations (5) and (9) one finds that the infinitesimal generator of rotations about the x_i-axis can be taken as

$$\mathcal{L}_i = \varepsilon_{ijk}x_j p_k - \frac{qg}{c}\frac{x_i}{|\mathbf{x}|}. \tag{20}$$

It can be readily verified that the operators (20) obey the usual commutation relations $[\mathcal{L}_i, \mathcal{L}_j] = i\hbar\varepsilon_{ijk}\mathcal{L}_k$ and generate a transformation group isomorphic to SO(3). In this case, $-(qg/c)\mathbf{x}/|\mathbf{x}|$ coincides with the angular momentum of the electromagnetic field computed by means of the standard expression (Jackson, 1975).

3. CONCLUDING REMARKS

As we have shown, the peculiar manner in which the interaction with a magnetic field is expressed in the Hamiltonian operator for a charged particle, imposes restrictions for the existence of the operators representing rigid motions. The derivations presented here follow from the basic question of what should be the effect of a rigid motion on the position and the kinematical momentum of a particle. Similar results hold in the more general case where there is an interaction with a non-Abelian gauge field or if one takes into account relativistic mechanics.

The results presented here are deeply related to those of Torres del Castillo (1999), where it is shown that the Schrödinger equation for a charged particle in a static magnetic field is equivalent to the Schrödinger equation for a free particle in a four-dimensional space with a suitably defined metric in such a way that the rigid motions that leave the magnetic field invariant give rise to symmetries of the four-dimensional metric.

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